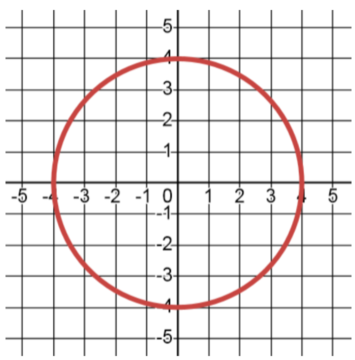
1

THE CIRCLE – CENTER OFF THE ORIGIN

You've graphed circles whose centers are at the origin in the Chapter *The Circle – Center at the Origin*. For example, the center of the circle



$x^2 + y^2 = 16$ is (0, 0), and its radius is 4. Our goal in this chapter is to allow the center of a circle to be somewhere else in the plane.

What kind of equation would result in such a circle? The example in the next section, $(x - 2)^2 + (y + 1)^2 = 25$, will provide us with one; but before we get there, we need to understand why this equation is likely a circle. If you square (expand) the binomials, note that you'll get an x^2 term, a y^2 term, some x's, some y's, and some numbers. The squared terms are the ones that lead us to believe that the graph of the equation is a circle, but we shall see.

❑ USING THE ANNIHILATOR METHOD

EXAMPLE 1: **Graph:** $\mathbf{(x - 2)^2 + (y + 1)^2 = 25}$

Solution: First, it might be clear (due to the –2 and the 1) that the center of the circle is not the origin. Second, our best guess right now is that the radius is 5 (since 5 is the positive square root of 25). Let's check out these theories.

It's hard to know what values of x we should choose to find our points to plot, but here's a neat trick to find four useful points.

We'll start with $\underline{x = 2}$ (this rids us of the first term), and then solve for y in the circle equation $(x - 2)^2 + (y + 1)^2 = 25$:

$$
\begin{aligned}
x = \mathbf{2} \quad &\Rightarrow \quad (\mathbf{2} - 2)^2 + (y + 1)^2 = 25 \\
&\Rightarrow \quad 0^2 + (y + 1)^2 = 25 \\
&\Rightarrow \quad (y + 1)^2 = 25 \\
&\Rightarrow \quad y + 1 = \pm 5 \qquad \text{[Don't forget: 25 has } \underline{2} \text{ square roots]} \\
&\Rightarrow \quad y = -1 \pm 5 \\
&\Rightarrow \quad y = 4 \text{ or } -6
\end{aligned}
$$

Since letting $x = 2$ produced two y-values, we have the two points **(2, 4)** and **(2, –6)** on our circle.

Next, we'll let $\underline{y = -1}$ (this annihilates the second term), and then solve for x in the circle equation $(x - 2)^2 + (y + 1)^2 = 25$:

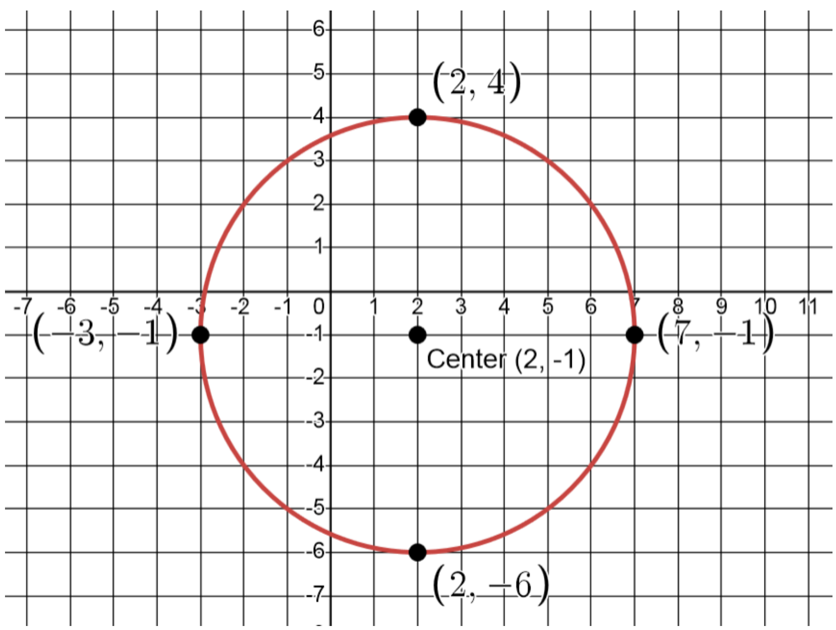
$$
\begin{aligned}
y = \mathbf{-1} \quad &\Rightarrow \quad (x - 2)^2 + (\mathbf{-1} + 1)^2 = 25 \\
&\Rightarrow \quad (x - 2)^2 + 0^2 = 25 \\
&\Rightarrow \quad (x - 2)^2 = 25 \\
&\Rightarrow \quad x - 2 = \pm 5 \\
&\Rightarrow \quad x = 2 \pm 5 \\
&\Rightarrow \quad x = 7 \text{ or } -3
\end{aligned}
$$

By letting $y = -1$, we determined that two more points on the circle are **(7, –1)** and **(–3, –1)**.

What have we done here? By cleverly choosing values of x and y (do you see how we chose them?), we have discovered the following four points on our circle:

$$(2, 4) \quad (2, -6) \quad (7, -1) \quad (-3, -1)$$

Let's plot these four points, connect them to make a circle, and then figure out the circle's center and radius.



Can you see that the center of the circle is the point (2, –1) and that the radius is 5? Let's summarize:

The circle $(x - 2)^2 + (y + 1)^2 = 25$ has its center at (2, –1) and has a radius of 5.

Homework

1. Find the **center** and **radius** of each circle by mimicking Example 1 (the *annihilator method*):

a. $x^2 + y^2 = 64$

b. $x^2 + y^2 = 24$

c. $x^2 + (y - 7)^2 = 4$

d. $(x - 3)^2 + y^2 = 5$

e. $(x + 1)^2 + (y + 8)^2 = 60$

f. $(x - 2)^2 + (y - 3)^2 = 99$

❑ *A Quicker Approach*

Look at the problem and the solution in Example 1. Do you see any connections here? The x-coordinate of the center (the 2) is the *opposite* of the number following the x in the circle equation. Also, the y-coordinate of the center (the –1) is the *opposite* of the number following the y in the circle equation. And last, the circle's radius, 5, is the positive square root of the 25 on the right side of the circle equation, just as we expected.

The following table shows the relationships between the equations of some circles and the circles' centers and radii. Study it carefully.

Equation of Circle	Center	Radius
$(x - 7)^2 + (y - 4)^2 = 121$	(7, 4)	11
$(x + 3)^2 + (y + 5)^2 = 49$	(–3, –5)	7
$(x - 1)^2 + (y + 12)^2 = 21$	(1, –12)	$\sqrt{21}$
$(x + 11)^2 + (y - 9)^2 = 93$	(–11, 9)	$\sqrt{93}$
$x^2 + (y - 8)^2 = 1$	(0, 8)	1
$(x + 13)^2 + y^2 = 50$	(–13, 0)	$5\sqrt{2}$

Equation of Circle	Center	Radius
$x^2 + y^2 = 108$	(0, 0)	$6\sqrt{3}$

Homework

2. Find the **center** and **radius** of each circle using the "shortcut" described in the chart above:

a. $(x + 5)^2 + (y - 3)^2 = 144$

b. $(x - 1)^2 + (y + 11)^2 = 48$

c. $(x + 7)^2 + (y + \pi)^2 = 50$

d. $(x - \sqrt{2})^2 + (y - 2.3)^2 = 1$

e. $(x + 3) + (y - 1) = 9$

f. $(x - 3)^2 + (y + 2) = 7$

3. Find the **equation** of the circle with the given *center* and *radius*:

a. C(0, 0) $r = 7$

b. C(0, 0) $r = \sqrt{10}$

c. C(3, 0) $r = 3$

d. C(0, 4) $r = 1$

e. C(0, –2) $r = \sqrt{3}$

f. C(–12, 0) $r = 12$

g. C(2, 7) $r = 10$

h. C(–1, –3) $r = 2\sqrt{3}$

i. C(3, –4) $r = 3\sqrt{5}$

j. C(–2, 9) $r = 5\sqrt{7}$

k. C(1, 2) $r = 0$

l. C(–3, 5) $r = -9$

❑ *A New Twist for the Circle*

Now for a tricky circle question:

Find the center and radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$.

This circle is not in the standard form we've been using to extract the center and radius. So, how in the heck do we convert this circle equation containing no parentheses into the proper form with parentheses? Any ideas before you read on?

EXAMPLE 2: **Graph the circle $x^2 + y^2 + 8x - 6y + 9 = 0$.**

Solution: Creating the graph will depend on finding the circle's center and radius. To calculate these, we need to convert the given equation of the circle into standard form.

Remember the "magic number" we used to solve quadratic equations by completing the square? We use the same trick here, except we will complete the square in both variables. Cool . . . we get to calculate two magic numbers!

Start with the given equation of the circle:

$$x^2 + y^2 + 8x - 6y + 9 = 0$$

Rearrange the terms, putting the x-terms next to each other and the y-terms next to each other:

$$x^2 + 8x + y^2 - 6y + 9 = 0$$

Take the constant 9 to the other side of the equation:

$$x^2 + 8x + y^2 - 6y = -9$$

Now calculate the two "magic numbers":

Half of 8 is 4, and $4^2 = \mathbf{16}$. This is the magic number for x.

Half of -6 is -3, and $(-3)^2 = \mathbf{9}$. This is the magic number for y.

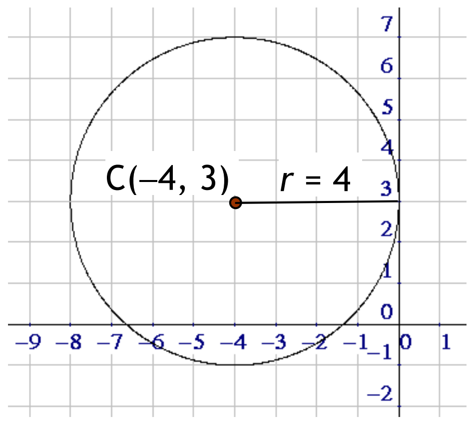
Now we add the magic numbers to both sides of the equation so that perfect square trinomials will be formed on the left side:

$$\underbrace{x^2 + 8x + \boxed{16}}_{\text{factorable}} + \underbrace{y^2 - 6y + \boxed{9}}_{\text{factorable}} = -9 + \boxed{16} + \boxed{9}$$

Now factor the first three terms, then factor the next three terms, and then do the arithmetic on the right side of the equation:

$$(x+4)^2 + (y-3)^2 = 16$$

We made it! Now that the circle equation is in standard form, we can read the center and radius directly from the equation. The center is (–4, 3) and the radius is 4 (the positive square root of 16). This info is just what we need to graph our circle:



Homework

4. Find the **center** and **radius** of each circle:

a. $x^2 + y^2 + 8x - 10y - 40 = 0$

b. $x^2 + y^2 - 16x + 8y + 31 = 0$

c. $x^2 + y^2 + 12x + 14y + 49 = 0$

d. $x^2 + y^2 + 14x + 24 = 0$

e. $x^2 + y^2 - 6y - 216 = 0$

f. $x^2 + y^2 + 8x + 6y + 10 = 0$

g. $x^2 + y^2 - 2x - 4y - 3 = 0$

h. $x^2 + y^2 + 4x - 10y + 12 = 0$

i. $x^2 + y^2 - 20x + 6y + 101 = 0$

> Note: If you come across a circle equation like
>
> $$3x^2 + 3y^2 + 24x - 15y + 6 = 0$$
>
> just divide each side of the equation by 3 (meaning divide every term by 3), and then proceed as usual.

j. $x^2 + y^2 + 2x - 4y + 25 = 0$

Review Problems

5. Describe the graph of each equation:

   a. $x^2 + y^2 = 1{,}000{,}000$ b. $x^2 + y^2 = 1$

   c. $x^2 + y^2 = 0$ d. $x^2 + y^2 = -9$

6. Consider the equation $(x - h)^2 + (y - k)^2 = C$. Describe the graph of this equation

   a. if $C > 0$ b. if $C = 0$ c. if $C < 0$

7. Find the center and radius of the circle $x^2 + y^2 = 20$.

8. Find the center and radius of the circle $x^2 + y^2 - 10x + 2y + 3 = 0$.

9. Matching:

____ $y = 3$	A. parabola
____ $2x^2 - 3y = 10$	B. circle
____ $x^2 + (y - 1)^2 = 2$	C. horizontal line
____ $x - y + 10 = 0$	D. non-horizontal line

10. True/False:

    a. Every circle has at least one intercept.

    b. The radius of the circle $x^2 + y^2 = 1$ is 1.

    c. The graph of $y = x^2 - 9x + 17$ is a parabola.

d. $x^2 + y^2 + 4 = 3$ is a circle.

e. $x^2 + y^2 = 1$ is called the *unit circle*.

f. The center of the circle $(x - 2)^2 + (y - 5)^2 = 10$ is $(-2, -5)$.

g. The radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$ is 4.

h. The graph of $10x^2 + 10y^2 = 37$ is a circle.

i. The graph of $10x^2 - 10y^2 = 37$ is a circle.

j. The graph of $10x^2 + 9y^2 = 37$ is a circle.

k. The area of the circle $x^2 + y^2 = 25$ is 25π.

Solutions

1. a. $C(0, 0)$ $r = 8$ b. $C(0, 0)$ $r = 2\sqrt{6}$

c. $C(0, 7)$ $r = 2$ d. $C(3, 0)$ $r = \sqrt{5}$

e. $C(-1, -8)$ $r = 2\sqrt{15}$ f. $C(2, 3)$ $r = 3\sqrt{11}$

2. a. $C(-5, 3)$ $r = 12$ b. $C(1, -11)$ $r = 4\sqrt{3}$

c. $C(-7, -\pi)$ $r = 5\sqrt{2}$ d. $C(\sqrt{2}, 2.3)$ $r = 1$

e. Trick: It's a line f. Trick: It's a parabola

3. a. $x^2 + y^2 = 49$ b. $x^2 + y^2 = 10$

c. $(x - 3)^2 + y^2 = 9$ d. $x^2 + (y - 4)^2 = 1$

e. $x^2 + (y + 2)^2 = 3$ f. $(x + 12)^2 + y^2 = 144$

g. $(x - 2)^2 + (y - 7)^2 = 100$ h. $(x + 1)^2 + (y + 3)^2 = 12$

i. $(x - 3)^2 + (y + 4)^2 = 45$ j. $(x + 2)^2 + (y - 9)^2 = 175$

k. Not a circle; just the point (1, 2)

l. Not a circle; in fact, no graph at all

4. a. $C(-4, 5)$ $r = 9$ b. $C(8, -4)$ $r = 7$

c. $C(-6, -7)$ $r = 6$ d. $C(-7, 0)$ $r = 5$

e. C(0, 3) $r = 15$

f. C(–4, –3) $r = \sqrt{15}$

g. C(1, 2) $r = 2\sqrt{2}$

h. C(–2, 5) $r = \sqrt{17}$

i. C(10, –3) $r = 2\sqrt{2}$

j. Not a circle

5. a. A circle with center at the origin and a radius of 1,000.
b. A circle with center at the origin and a radius of 1.
c. The point (0, 0), and that's it.
d. Since the sum of two squares is never negative, there is NO graph.

6. a. If $C > 0$, the graph is a circle with center at the origin and radius $\sqrt{C}$.
b. If $C = 0$, the graph is just the single point (h, k); i.e., the "center" of a phantom circle.
c. If $C < 0$, the graph is empty (there's no graph at all).

7. C(0, 0); $r = 2\sqrt{5}$

8. C(5, –1); $r = \sqrt{23}$

9. C, A, B, D

10. a. F b. T c. T d. F e. T f. F
g. T h. T i. F j. F k. T

"The educated differ from
the uneducated
as much as the living from
the dead."

Αριστοτλε (Aristotle)